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COUPLED BENDING-TORSIONAL DYNAMIC STIFFNESS MATRIX OF AN AXIALLY LOADED TIMOSHENKO BEAM ELEMENT

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Abstract—Analytical expressions for the coupled bending-torsional dynamic stiffness matrix terms of an axially loaded uniform Timoshenko beam element are derived in an exact sense by solving the governing differential equations of motion of the element. The symbolic computing package REDUCE has been used to generate an analytical expression for each of the dynamic stiffness terms in a concise form. For check purposes, numerical values of the dynamic stiffness matrix terms were obtained using the derived explicit expressions as well as by an alternative nonanalytical method based on matrix inversions and matrix multiplications. Stiffnesses obtained from both methods agreed with each other to machine accuracy. Application of the developed theory is discussed with particular reference to an established algorithm. The influence of axial force, shear deformation and rotatory inertia on the natural frequencies of a bending-torsion coupled beam with cantilever end-conditions is demonstrated by numerical results. Such results are not generally available in the literature. Therefore, results obtained by partially restricting the present theory are compared with the existing literature wherever possible. The results indicate that the method is accurate and efficient.

1. INTRODUCTION

An elegant and powerful means of solving vibration problems in structural engineering, particularly when higher natural frequencies and better accuracies are required, is to use the dynamic (i.e. frequency dependent) stiffness matrix method. This method is often referred to as an exact method because, unlike the traditional finite element and other approximate methods, it allows an infinite number of natural frequencies and normal modes of a vibrating structure to be accounted for, without any loss of accuracy. The method gives accurate results irrespective of the number of elements considered in the analysis, because all assumptions made in the method are within the limits of the classical theory of the differential equations of motion of the element.

The historical development of the dynamic stiffness matrix of a beam element occurred in stages. First the dynamic stiffness matrix of a Bernoulli–Euler beam was developed by Kolousek (1941, 1943) who later included it in a text book (1973). Williams and Wittrick (1973) used such theory to investigate the vibration characteristics of offshore dolphin structures accurately and with great computational efficiency. Also Mohsin and Sadek (1968) took the simple beam theory further by taking steps towards the development of the dynamic stiffness matrix of an axially loaded Bernoulli–Euler beam, by considering the effect of a static axial load. Afterwards, the development of the dynamic stiffness formulation for a Bernoulli–Euler beam with the effects of shear deformation and rotatory inertia included, i.e. for a Timoshenko beam, but without the effect of axial force, involved much effort by Cheng (1970) and Wang and Kinsman (1971). Next, the dynamic stiffness matrix of an axially loaded Timoshenko beam was developed, e.g. by Howson and Williams (1973) and by Cheng and Tseng (1973). Subsequent developments of the method include notable contributions by Akesson (1976), Richards and Leung (1977), Lunden and Akesson (1983), Banerjee and Williams (1985), Williams and Kennedy (1987) and Issa (1988). The culmination of several years of research on the dynamic stiffness development of beam theory resulted in its computer implementation, e.g. to give the BUNVIS-RG (Anderson *et al.*, 1986) code, which is now a quite well-established research tool.

The present paper addresses coupled bending-torsional vibration of (statically) axially loaded beams within the context of the dynamic stiffness matrix method of analysing structures. Such coupled vibration is particularly important to the aerospace industry because of its aeroelastic applications. As with simple beams, for which the bending and torsional motions are uncoupled, the development of the dynamic stiffness matrix for a coupled beam has taken place in stages. The coupled bending-torsional dynamic stiffness matrix of a Bernoulli–Euler beam was developed by Hallauer and Liu (1982), Friberg (1983) and Banerjee (1989, 1991). The authors of these papers ignored the effects of shear deformation, rotatory inertia and axial force in their theory. The method of Banerjee (1989, 1991) differed from those of Hallauer and Liu (1982) and Friberg (1983) because Banerjee generated explicit algebraic expressions for the (scalar) terms of the dynamic stiffness matrix, whereas the other authors obtained the dynamic stiffness matrix numerically. Banerjee's stiffness expressions are particularly useful when some but not all the stiffnesses are needed, for example, when solving the aeroelastic problems of a cantilever wing. Later, the dynamic stiffness matrices were developed for an axially loaded bending-torsion coupled Bernoulli–Euler beam (Banerjee and Fisher, 1992) and for a bending-torsion coupled Timoshenko beam (Banerjee and Williams, 1992). The authors of these papers obtained explicit algebraic expressions for dynamic stiffness terms by performing symbolic computations, using the well known package REDUCE (Rayna, 1986). They also showed that programming the explicit expressions for the dynamic stiffness matrix terms saves significant computation time when compared with numerical methods, see Table 1 of Banerjee and Williams (1992), where there was at least an eight-fold advantage in computer time for the particular problem type investigated.

The development of the dynamic stiffness matrix of a Vlasov beam is very recent and appears to have been reported by only Friberg (1985) and Leung (1991, 1992). Both of these authors solved this complex problem in a novel and elegant way, but obtained the dynamic stiffness matrix numerically. No attempts appear to have been made to obtain explicit algebraic expressions for the terms of this dynamic stiffness matrix, which would undoubtedly require extensive algebraic manipulation, e.g. inverting a 6×6 matrix algebraically. It should be noted that the Vlasov theory used by Friberg (1985) and Leung (1991) accounted for rotatory inertia, axial force and warping stiffness of the beam, but ignored the important effect of shear deformation completely.

The present paper develops the explicit dynamic stiffness matrix of an axially loaded, bending-torsion coupled beam with the effects of the rotatory inertia and the shear deformation included. Thus it combines and unifies the earlier theories of Banerjee (1989), Banerjee and Fisher (1992) and Banerjee and Williams (1992) to give explicit analytical expressions for every term of the coupled bending-torsional dynamic stiffness matrix of an axially loaded uniform Timoshenko beam. However, the effect of the warping stiffness is considered to be small and is neglected in the analysis. The application of this dynamic stiffness matrix to obtain natural frequencies and normal mode shapes is demonstrated by using the established algorithm of Wittrick and Williams (1971) to ensure convergence on all required natural frequencies. (This algorithm has featured frequently in the literature, e.g. see the survey in Williams and Wittrick, 1983). An illustrative example on the application of the theory is given for a bending-torsion coupled beam with a thin-walled semi-circular cross-section (Friberg, 1985). The numerical values of the frequency-dependent dynamic stiffness terms are given for two representative frequencies to ten figure accuracy as a comparator for other workers who wish to use the stiffness expressions or to compare with alternative methods. The natural frequencies are also calculated for cantilever end-conditions of the beam and compared with those available in the literature.

2. THEORY

2.1. *Equations of motion and end-conditions*

A straight uniform beam element of length L and of aerofoil cross-section is shown in Fig. 1, with the mass axis and the elastic axis (i.e. the loci of the mass centre and the shear centre of the cross section) being separated by a distance x_α. In the right-handed coordinate system of Fig. 1, the elastic axis, which is assumed to coincide with the Y-axis, is permitted flexural translation $h(y, t)$ in the Z-direction and torsional rotation $\psi(y, t)$ about the Y-axis, where y and t denote distance from the origin and time, respectively. A constant compressive axial load P is assumed to act through the centroid (mass centre) of the cross-section. P can be positive or negative, so that tension is included.

The governing partial differential equations of motion for the coupled bending-torsional free natural vibration of the axially loaded Timoshenko beam shown in Fig. 1 are given by (for details of the derivation, see the Appendix)

$$EI\theta'' + kAG(h' - \theta) - \rho I\ddot{\theta} = 0, \tag{1}$$

$$kAG(h'' - \theta') - P(h'' - x_\alpha \psi'') - m(\ddot{h} - x_\alpha \ddot{\psi}) = 0, \tag{2}$$

$$GJ\psi'' - P\{(I_\alpha/m)\psi'' - x_\alpha h''\} - I_\alpha \ddot{\psi} + m x_\alpha \ddot{h} = 0, \tag{3}$$

where: E is the Young's modulus, G is the shear modulus and ρ is the density of the material; EI, GJ and kAG are, respectively, the bending, torsional and shear rigidities of the beam; I is the second moment of area of the beam cross-section about the X-axis, k is the section shape factor, A is the cross-section area, m $(= \rho A)$ is the mass per unit length, I_α is the polar mass moment of inertia per unit length about the Y-axis (i.e. an axis through the shear centre), θ is the angle of rotation in radians of the cross-section due to bending alone (so that the total slope h' equals the sum of slopes due to bending and due to shear deformation) and primes and dots denote differentiation with respect to position y and time t, respectively.

Equations (1)–(3) together with appropriate end conditions completely define the coupled bending-torsional free vibration of an axially loaded uniform Timoshenko beam.

If a sinusoïdal variation of h, θ and ψ, with circular frequency ω, is assumed, then

$$\left.\begin{aligned} h(y, t) &= H(y) \sin \omega t \\ \theta(y, t) &= \Theta(y) \sin \omega t \\ \psi(y, t) &= \Psi(y) \sin \omega t \end{aligned}\right\}, \tag{4}$$

where $H(y)$, $\Theta(y)$ and $\Psi(y)$ are the amplitudes of the sinusoidally varying vertical displacement, bending rotation and twist, respectively.

Substituting eqn (4) into eqns (1)–(3) gives

$$EI\Theta'' + kAG(H' - \Theta) + \rho I\omega^2 \Theta = 0, \tag{5}$$

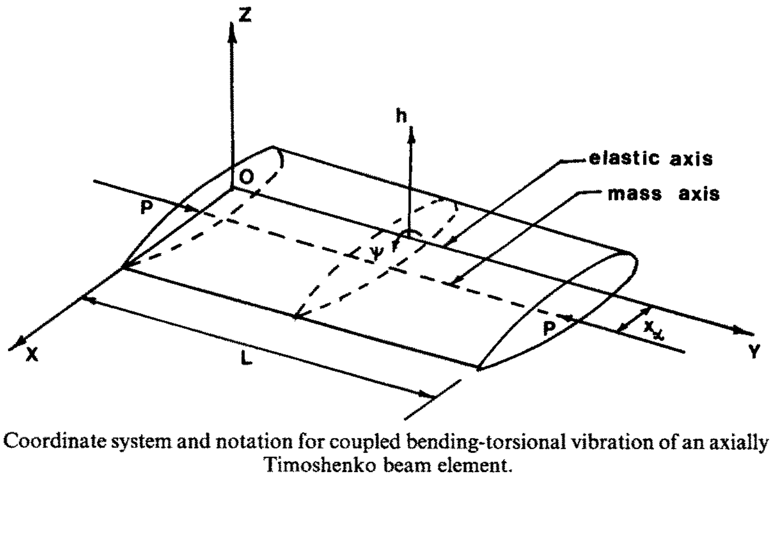

Fig. 1. Coordinate system and notation for coupled bending-torsional vibration of an axially loaded Timoshenko beam element.

$$kAG(H''-\Theta')-P(H''-x_\alpha\Psi'')+m\omega^2H-m\omega^2x_\alpha\Psi = 0, \tag{6}$$

$$GJ\Psi''-P\{(I_\alpha/m)\Psi''-x_\alpha H''\}+I_\alpha\omega^2\Psi-m\omega^2x_\alpha H = 0. \tag{7}$$

By extensive algebraic manipulation, eqns (5)–(7) can be combined into one equation by eliminating all but one of the three variables H, Θ and Ψ, to give:

$$(D^6+\bar{a}D^4-\bar{b}D^2-\bar{c})W = 0, \tag{8}$$

where

$$W = H, \Theta \text{ or } \Psi, \tag{9}$$

$$\left.\begin{aligned} D &= d/d\xi \\ \xi &= y/L \end{aligned}\right\}, \tag{10}$$

and

$$\left.\begin{aligned} \bar{a} &= b^2r^2+\frac{\{a^2b^2(1-2c^2p^2s^2)-a^2c^2p^4+b^2(p^2+b^2s^2)\}}{\{b^2(1-p^2s^2)-a^2p^2(1-c^2p^2s^2)\}} \\ \bar{b} &= \frac{\{b^4(1-b^2r^2s^2)-a^2b^4c^2s^2(1-2p^2r^2)-a^2b^2(2c^2p^2+b^2r^2)\}}{\{b^2(1-p^2s^2)-a^2p^2(1-c^2p^2s^2)\}} \\ \bar{c} &= \frac{a^2b^4c^2(1-b^2r^2s^2)}{\{b^2(1-p^2s^2)-a^2p^2(1-c^2p^2s^2)\}} \end{aligned}\right\}, \tag{11}$$

with

$$\left.\begin{aligned} a^2 &= I_\alpha\omega^2L^2/GJ \\ b^2 &= m\omega^2L^4/EI \\ c^2 &= 1-mx_\alpha^2/I_\alpha = I_G/I_\alpha \\ p^2 &= PL^2/EI \\ r^2 &= I/AL^2 \\ s^2 &= EI/kAGL^2 \end{aligned}\right\}, \tag{12}$$

where I_G is the polar mass moment of inertia per unit length about an axis through the centroid.

The solution of the differential eqn (8) is (Banerjee, 1989)

$$W(\xi) = C_1^*\cosh\alpha\xi+C_2^*\sinh\alpha\xi+C_3^*\cos\beta\xi+C_4^*\sin\beta\xi+C_5^*\cos\gamma\xi+C_6^*\sin\gamma\xi, \tag{13}$$

where $C_1^*-C_6^*$ are constants and

$$\left.\begin{aligned} \alpha &= [2(q/3)^{1/2}\cos(\phi/3)-\bar{a}/3]^{1/2} \\ \beta &= [2(q/3)^{1/2}\cos\{(\pi-\phi)/3\}+\bar{a}/3]^{1/2} \\ \gamma &= [2(q/3)^{1/2}\cos\{(\pi+\phi)/3\}+\bar{a}/3]^{1/2} \end{aligned}\right\}, \tag{14}$$

with

$$q = \bar{b}+\bar{a}^2/3$$

and

$$\phi = \cos^{-1}[(27\bar{c}-9\bar{a}\bar{b}-2\bar{a}^3)/\{2(\bar{a}^2+3\bar{b})^{3/2}\}] \tag{15}$$

Equation (13) represents the solution for the bending displacement $H(\xi)$, bending rotation $\Theta(\xi)$ and torsional rotation $\Psi(\xi)$. Thus

$$H(\xi) = A_1 \cosh \alpha\xi + A_2 \sinh \alpha\xi + A_3 \cos \beta\xi + A_4 \sin \beta\xi + A_5 \cos \gamma\xi + A_6 \sin \gamma\xi, \quad (16)$$

$$\Theta(\xi) = B_1 \sinh \alpha\xi + B_2 \cosh \alpha\xi + B_3 \sin \beta\xi + B_4 \cos \beta\xi + B_5 \sin \gamma\xi + B_6 \cos \gamma\xi, \quad (17)$$

$$\Psi(\xi) = C_1 \cosh \alpha\xi + C_2 \sinh \alpha\xi + C_3 \cos \beta\xi + C_4 \sin \beta\xi + C_5 \cos \gamma\xi + C_6 \sin \gamma\xi, \quad (18)$$

where $A_1 - A_6$, $B_1 - B_6$ and $C_1 - C_6$ are the three different sets of constants.

Substituting eqns (16) and (17) into eqn (5) shows that

$$\left.\begin{array}{lll} B_1 = (\bar{\alpha}/L)A_1; & B_3 = -(\bar{\beta}/L)A_3; & B_5 = -(\bar{\gamma}/L)A_5 \\ B_2 = (\bar{\alpha}/L)A_2; & B_4 = (\bar{\beta}/L)A_4; & B_6 = (\bar{\gamma}/L)A_6 \end{array}\right\}, \quad (19)$$

where

$$\left.\begin{array}{l} \bar{\alpha} = \alpha/(1 - b^2r^2s^2 - \alpha^2 s^2) \\ \bar{\beta} = \beta/(1 - b^2r^2s^2 + \beta^2 s^2) \\ \bar{\gamma} = \gamma/(1 - b^2r^2s^2 + \gamma^2 s^2) \end{array}\right\}. \quad (20)$$

Then substituting eqns (16) and (18) into eqn (7) gives

$$\left.\begin{array}{lll} C_1 = (k_\alpha/x_\alpha)A_1; & C_3 = (k_\beta/x_\alpha)A_3; & C_5 = (k_\gamma/x_\alpha)A_5 \\ C_2 = (k_\alpha/x_\alpha)A_2; & C_4 = (k_\beta/x_\alpha)A_4; & C_6 = (k_\gamma/x_\alpha)A_6 \end{array}\right\}, \quad (21)$$

where

$$\left.\begin{array}{l} k_\alpha = a^2(1-c^2)(b^2 - p^2\alpha^2)/\{a^2(b^2 - p^2\alpha^2) + b^2\alpha^2\} \\ k_\beta = a^2(1-c^2)(b^2 + p^2\beta^2)/\{a^2(b^2 + p^2\beta^2) - b^2\beta^2\} \\ k_\gamma = a^2(1-c^2)(b^2 + p^2\gamma^2)/\{a^2(b^2 + p^2\gamma^2) - b^2\gamma^2\} \end{array}\right\}, \quad (22)$$

Following the sign convention of Fig. 2, the expressions for the bending moment $M(\xi)$, the transverse force $S(\xi)$ and the torque $T(\xi)$ are obtained from eqns (16)–(18), after some simplification, as

$$M(\xi) = -(EI/L)\frac{d\Theta}{d\xi} = -(EI/L^2)\{A_1\alpha\bar{\alpha}\cosh\alpha\xi + A_2\alpha\bar{\alpha}\sinh\alpha\xi$$
$$-A_3\beta\bar{\beta}\cos\beta\xi - A_4\beta\bar{\beta}\sin\beta\xi - A_5\gamma\bar{\gamma}\cos\gamma\xi - A_6\gamma\bar{\gamma}\sin\gamma\xi\}, \quad (23)$$

$$S(\xi) = (EI/L^3)\left[L\frac{d^2\Theta}{d\xi^2} + p^2\left(\frac{dH}{d\xi} - x_\alpha\frac{d\Psi}{d\xi}\right) + b^2r^2\Theta L\right]$$
$$= (EI/L^3)\{A_1\bar{\alpha}g_\alpha \sinh\alpha\xi + A_2\bar{\alpha}g_\alpha\cosh\alpha\xi + A_3\bar{\beta}g_\beta\sin\beta\xi$$
$$-A_4\bar{\beta}g_\beta\cos\beta\xi + A_5\bar{\gamma}g_\gamma\sin\gamma\xi - A_6\bar{\gamma}g_\gamma\cos\gamma\xi\}, \quad (24)$$

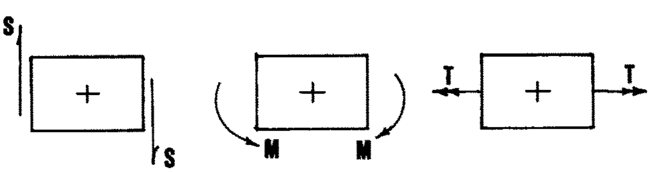

Fig. 2. Sign convention for positive transverse (shear) force (S), bending moment (M) and torque (T).

and

$$T(\xi) = (GJ/L)\left[(1-p^2a^2/b^2)\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}+\{p^2a^2(1-c^2)/(x_\alpha b^2)\}\frac{\mathrm{d}H}{\mathrm{d}\xi}\right]$$
$$= (GJ/L)\{A_1(\alpha e_\alpha/x_\alpha)\sinh\alpha\xi + A_2(\alpha e_\alpha/x_\alpha)\cosh\alpha\xi - A_3(\beta e_\beta/x_\alpha)\sin\beta\xi + A_4(\beta e_\beta/x_\alpha)\cos\beta\xi - A_5(\gamma e_\gamma/x_\alpha)\sin\gamma\xi + A_6(\gamma e_\gamma/x_\alpha)\cos\gamma\xi\}, \quad (25)$$

where

$$\left.\begin{aligned} g_\alpha &= \alpha^2+b^2r^2+p^2(1-k_\alpha)\alpha/\bar{\alpha} \\ g_\beta &= \beta^2-b^2r^2-p^2(1-k_\beta)\beta/\bar{\beta} \\ g_\gamma &= \gamma^2-b^2r^2-p^2(1-k_\gamma)\gamma/\bar{\gamma} \end{aligned}\right\}, \quad (26)$$

and

$$\left.\begin{aligned} e_\alpha &= (1-a^2p^2/b^2)k_\alpha+a^2p^2(1-c^2)/b^2 \\ e_\beta &= (1-a^2p^2/b^2)k_\beta+a^2p^2(1-c^2)/b^2 \\ e_\gamma &= (1-a^2p^2/b^2)k_\gamma+a^2p^2(1-c^2)/b^2 \end{aligned}\right\}, \quad (27)$$

The end conditions for displacements and forces of the beam element (see Fig. 3) are, respectively

displacements:

$$\left.\begin{aligned} &\text{at end 1 (i.e. } \xi = 0): H = H_1, \Theta = \Theta_1 \text{ and } \Psi = \Psi_1 \\ &\text{at end 2 (i.e. } \xi = 1): H = H_2, \Theta = \Theta_2 \text{ and } \Psi = \Psi_2 \end{aligned}\right\}, \quad (28)$$

forces

$$\left.\begin{aligned} &\text{at end 1 (i.e. } \xi = 0): S = S_1, M = M_1 \text{ and } T = -T_1 \\ &\text{at end 2 (i.e. } \xi = 1): S = -S_2, M = -M_2 \text{ and } T = T_2 \end{aligned}\right\}. \quad (29)$$

2.2. *Derivation of the dynamic stiffness matrix*

The dynamic stiffness matrix which relates the amplitudes of the sinusoidally varying forces to the corresponding displacement amplitudes can now be derived with the help of eqns (16)–(18), (23)–(25), (28) and (29) as follows.

Substituting eqns (28) and (19) in eqns (16)–(18) gives

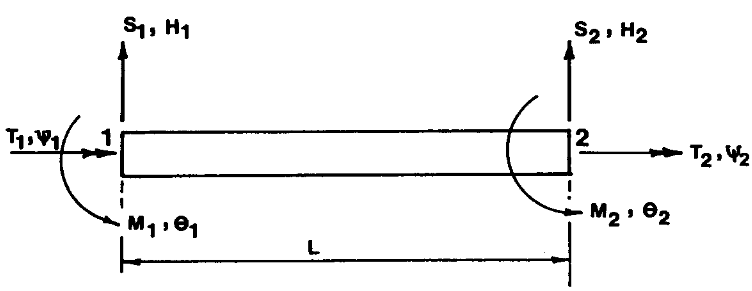

Fig. 3. End conditions for forces and displacements of the beam element.

$$\begin{bmatrix} H_1 \\ \Theta_1 \\ \Psi_1 \\ H_2 \\ \Theta_2 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \bar{\alpha}/L & 0 & \bar{\beta}/L & 0 & \bar{\gamma}/L \\ k_\alpha/x_\alpha & 0 & k_\beta/x_\alpha & 0 & k_\gamma/x_\alpha & 0 \\ C_{h\alpha} & S_{h\alpha} & C_\beta & S_\beta & C_\gamma & S_\gamma \\ \bar{\alpha}S_{h\alpha}/L & \bar{\alpha}C_{h\alpha}/L & -\bar{\beta}S_\beta/L & \bar{\beta}C_\beta/L & -\bar{\gamma}S_\gamma/L & \bar{\gamma}C_\gamma/L \\ k_\alpha C_{h\alpha}/x_\alpha & k_\alpha S_{h\alpha}/x_\alpha & k_\beta C_\beta/x_\alpha & k_\beta S_\beta/x_\alpha & k_\gamma C_\gamma/x_\alpha & k_\gamma S_\gamma/x_\alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix}, \quad (30)$$

i.e.

$$\mathbf{U} = \mathbf{BA}, \quad (31)$$

where

$$\left.\begin{aligned} C_{h\alpha} &= \cosh\alpha; \quad C_\beta = \cos\beta; \quad C_\gamma = \cos\gamma \\ S_{h\alpha} &= \sinh\alpha; \quad S_\beta = \sin\beta; \quad S_\gamma = \sin\gamma \end{aligned}\right\}. \quad (32)$$

Substituting eqn (29) in eqns (23)–(25) gives:

$$\begin{bmatrix} S_1 \\ M_1 \\ T_1 \\ S_2 \\ M_2 \\ T_2 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & W_3\bar{\alpha}g_\alpha & 0 & -W_3\bar{\beta}g_\beta & 0 & -W_3\bar{\gamma}g_\gamma \\ -W_2\alpha\bar{\alpha} & 0 & W_2\beta\bar{\beta} & 0 & W_2\gamma\bar{\gamma} & 0 \\ 0 & -W_1\alpha e_\alpha/x_\alpha & 0 & -W_1\beta e_\beta/x_\alpha & 0 & -W_1\gamma e_\gamma/x_\alpha \\ -W_3\bar{\alpha}S_{h\alpha}g_\alpha & -W_3\bar{\alpha}C_{h\alpha}g_\alpha & -W_3\bar{\beta}S_\beta g_\beta & W_3\bar{\beta}C_\beta g_\beta & -W_3\bar{\gamma}S_\gamma g_\gamma & W_3\bar{\gamma}C_\gamma g_\gamma \\ W_2\alpha\bar{\alpha}C_{h\alpha} & W_2\alpha\bar{\alpha}S_{h\alpha} & -W_2\beta\bar{\beta}C_\beta & -W_2\beta\bar{\beta}S_\beta & -W_2\gamma\bar{\gamma}C_\gamma & -W_2\gamma\bar{\gamma}S_\gamma \\ W_1\alpha e_\alpha S_{h\alpha}/x_\alpha & W_1\alpha e_\alpha C_{h\alpha}/x_\alpha & -W_1\beta e_\beta S_\beta/x_\alpha & W_1\beta e_\beta C_\beta/x_\alpha & -W_1\gamma e_\gamma S_\gamma/x_\alpha & W_1\gamma e_\gamma C_\gamma/x_\alpha \end{bmatrix}$$

$$\times \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix}, \quad (33)$$

i.e.

$$\mathbf{F} = \mathbf{DA}, \quad (34)$$

where

$$W_1 = GJ/L; \quad W_2 = EI/L^2; \quad W_3 = EI/L^3. \quad (35)$$

Equations (31) and (34) give

$$\mathbf{F} = \mathbf{KU}, \quad (36)$$

i.e.

$$\begin{bmatrix} S_1 \\ M_1 \\ T_1 \\ S_2 \\ M_2 \\ T_2 \end{bmatrix}, = \begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} & K_{1,5} & K_{1,6} \\ & K_{2,2} & K_{2,3} & K_{2,4} & K_{2,5} & K_{2,6} \\ & & K_{3,3} & K_{3,4} & K_{3,5} & K_{3,6} \\ & \text{symmetric} & & K_{4,4} & K_{4,5} & K_{4,6} \\ & & & & K_{5,5} & K_{5,6} \\ & & & & & K_{6,6} \end{bmatrix} \begin{bmatrix} H_1 \\ \Theta_1 \\ \Psi_1 \\ H_2 \\ \Theta_2 \\ \Psi_2 \end{bmatrix}, \tag{37}$$

where

$$\mathbf{K} = \mathbf{D}\mathbf{B}^{-1}, \tag{38}$$

is the required stiffness matrix.

2.3. *Explicit expressions for the terms of the dynamic stiffness matrix*

The task of inverting the **B** matrix algebraically and then premultiplying the inverted matrix by the **D** matrix was quite formidable and became possible due to the recent development in symbolic computing (Fitch, 1985). Thus it was greatly assisted by the symbolic computing package REDUCE (Rayna, 1986) when generating and, more importantly, when simplifying the explicit expressions for the terms of the dynamic stiffness matrix. Such expressions are particularly useful when some but not all of the stiffness coefficients are needed. The derived expressions are presented in concise form in eqns (51), if appropriate substitutions are made from eqns (39)–(50) and (52)–(53), as follows.

Let the following variables be introduced:

$$\mu_1 = k_\alpha - k_\beta; \quad \mu_2 = k_\beta - k_\gamma; \quad \mu_3 = k_\gamma - k_\alpha; \tag{39}$$

$$\nu_1 = g_\alpha + g_\beta; \quad \nu_2 = g_\beta + g_\gamma; \quad \nu_3 = g_\gamma + g_\alpha; \tag{40}$$

$$\delta_1 = g_\alpha - g_\beta; \quad \delta_2 = g_\beta - g_\gamma; \quad \delta_3 = g_\gamma - g_\alpha; \tag{41}$$

$$\eta_1 = \alpha\bar{\alpha} + \beta\bar{\beta}; \quad \eta_2 = \beta\bar{\beta} + \gamma\bar{\gamma}; \quad \eta_3 = \gamma\bar{\gamma} + \alpha\bar{\alpha}; \tag{42}$$

$$\zeta_1 = \alpha\bar{\alpha} - \beta\bar{\beta}; \quad \zeta_2 = \beta\bar{\beta} - \gamma\bar{\gamma}; \quad \zeta_3 = \gamma\bar{\gamma} - \alpha\bar{\alpha}; \tag{43}$$

$$\xi_1 = \alpha\bar{\beta}e_\alpha - \beta\bar{\alpha}e_\beta; \quad \xi_2 = \beta\bar{\gamma}e_\beta - \gamma\bar{\beta}e_\gamma; \quad \xi_3 = \gamma\bar{\alpha}e_\gamma - \alpha\bar{\gamma}e_\alpha; \tag{44}$$

$$\varepsilon_1 = \alpha\bar{\alpha}\mu_2 - \beta\bar{\beta}\mu_3 - \gamma\bar{\gamma}\mu_1; \quad \varepsilon_2 = \mu_1\nu_1 - \mu_2\delta_2 - \mu_3\nu_3; \quad \varepsilon_3 = \mu_1\xi_1\bar{\gamma} + \mu_2\xi_2\bar{\alpha} + \mu_3\xi_3\bar{\beta}; \tag{45}$$

$$\kappa_1 = \mu_1\nu_3 - \mu_3\nu_1; \quad \kappa_2 = \mu_2\nu_1 - \mu_1\delta_2; \quad \kappa_3 = \mu_2\nu_3 + \mu_3\delta_2; \tag{46}$$

$$\bar{\kappa}_1 = \mu_1\xi_3\bar{\beta} + \mu_3\xi_1\bar{\gamma}; \quad \bar{\kappa}_2 = \mu_2\xi_1\bar{\gamma} + \mu_1\xi_2\bar{\alpha}; \quad \bar{\kappa}_3 = \mu_3\xi_2\bar{\alpha} + \mu_2\xi_3\bar{\beta}; \tag{47}$$

$$\lambda_1 = \bar{\beta}\bar{\gamma}\mu_1\mu_3; \quad \lambda_2 = \bar{\alpha}\bar{\gamma}\mu_1\mu_2; \quad \lambda_3 = \bar{\alpha}\bar{\beta}\mu_2\mu_3; \tag{48}$$

$$\chi_1 = \bar{\gamma}^2\nu_3 k_\beta\mu_1 - \bar{\beta}^2\nu_1 k_\gamma\mu_3; \quad \chi_2 = \bar{\alpha}^2\nu_1 k_\gamma\mu_2 + \bar{\gamma}^2\delta_2 k_\alpha\mu_1; \quad \chi_3 = \bar{\alpha}^2\nu_3 k_\beta\mu_2 - \bar{\beta}^2\delta_2 k_\alpha\mu_3; \tag{49}$$

$$\tau_1 = \mu_1\nu_3 k_\gamma - \mu_3\nu_1 k_\beta; \quad \tau_2 = \mu_2\nu_1 k_\alpha - \mu_1\delta_2 k_\gamma; \quad \tau_3 = \mu_2\nu_3 k_\alpha + \mu_3\delta_2 k_\beta; \tag{50}$$

where: α, β and γ; $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$; k_α, k_β and k_γ; g_α, g_β and g_γ; and e_α, e_β and e_γ are, respectively, given by eqns (14), (20), (22), (26) and (27).

Then the terms of the required stiffness matrix, see eqn (37), can be found from

$$\left.\begin{aligned}
K_{1,1} &= K_{4,4} = (EI/L^3)(\Phi_1/\Delta)\\
K_{1,2} &= -K_{4,5} = (EI/L^2)(\Phi_2/\Delta)\\
K_{1,3} &= K_{4,6} = (x_\alpha EI/L^3)(\Phi_3/\Delta)\\
K_{1,4} &= (EI/L^3)(\Phi_4/\Delta)\\
K_{1,5} &= -K_{2,4} = (EI/L^2)(\Phi_5/\Delta)\\
K_{1,6} &= K_{3,4} = (x_\alpha EI/L^3)(\Phi_6/\Delta)\\
K_{2,2} &= K_{5,5} = (EI/L)(\Phi_7/\Delta)\\
K_{2,3} &= -K_{5,6} = (x_\alpha EI/L^2)(\Phi_8/\Delta)\\
K_{2,5} &= (EI/L)(\Phi_9/\Delta)\\
K_{2,6} &= -K_{3,5} = (x_\alpha EI/L^2)(\Phi_{10}/\Delta)\\
K_{3,3} &= K_{6,6} = (GJ/L)(\Phi_{11}/\Delta)\\
K_{3,6} &= (GJ/L)(\Phi_{12}/\Delta)
\end{aligned}\right\}, \tag{51}$$

where

$$\left.\begin{aligned}
\Phi_1 &= -\bar\beta\chi_2 C_\beta S_\gamma S_{h\alpha} - \bar\alpha\chi_1 S_\beta S_\gamma C_{h\alpha} + \bar\gamma\chi_3 S_\beta C_\gamma S_{h\alpha} + \bar\alpha\bar\beta\bar\gamma(\tau_1 C_{h\alpha} - \tau_2 C_\beta + \tau_3 C_\gamma)\\
&\quad + \bar\alpha\bar\beta\bar\gamma(\mu_3 k_\beta v_3 + \mu_2 k_\alpha \delta_2 - \mu_1 k_\gamma v_1) C_\beta C_\gamma C_{h\alpha}\\
\Phi_2 &= (\bar\alpha^2 g_\alpha \mu_2^2 + \bar\beta^2 g_\beta \mu_3^2 + \bar\gamma^2 g_\gamma \mu_1^2) S_\beta S_\gamma S_{h\alpha} + \lambda_2\delta_3 S_\beta(1 - C_\gamma C_{h\alpha}) - \lambda_3\delta_1 S_\gamma(1 - C_\beta C_{h\alpha})\\
&\quad + \lambda_1 v_2 S_{h\alpha}(1 - C_\beta C_\gamma)\\
\Phi_3 &= \bar\beta(\bar\alpha^2\mu_2 v_1 + \bar\gamma^2\mu_1\delta_2) C_\beta S_\gamma S_{h\alpha} + \bar\alpha(\bar\gamma^2\mu_1 v_3 - \bar\beta^2\mu_3 v_1) S_\beta S_\gamma C_{h\alpha}\\
&\quad - \bar\gamma(\bar\alpha^2\mu_2 v_3 - \bar\beta^2\mu_3\delta_2) S_\beta C_\gamma S_{h\alpha} + \bar\alpha\bar\beta\bar\gamma(\varepsilon_2 C_\beta C_\gamma C_{h\alpha} - \kappa_1 C_{h\alpha} + \kappa_2 C_\beta - \kappa_3 C_\gamma)\\
\Phi_4 &= \bar\alpha\bar\beta\bar\gamma(\tau_1 C_\gamma C_\beta - \tau_2 C_\gamma C_{h\alpha} + \tau_3 C_\beta C_{h\alpha} - \mu_1 v_1 k_\gamma + \mu_2\delta_2 k_\alpha + \mu_3 v_3 k_\beta)\\
&\quad + \bar\alpha\chi_1 S_\beta S_\gamma + \bar\beta\chi_2 S_\gamma S_{h\alpha} - \bar\gamma\chi_3 S_\beta S_{h\alpha}\\
\Phi_5 &= \lambda_1\delta_2 S_{h\alpha}(C_\beta - C_\gamma) - \lambda_2 v_3 S_\beta(C_{h\alpha} - C_\gamma) - \lambda_3 v_1 S_\gamma(C_{h\alpha} - C_\beta)\\
\Phi_6 &= \bar\alpha\bar\beta\bar\gamma(\varepsilon_2 - \kappa_3 C_\beta C_{h\alpha} + \kappa_2 C_\gamma C_{h\alpha} - \kappa_1 C_\beta C_\gamma) + \bar\alpha(\bar\beta^2\mu_3 v_1 - \bar\gamma^2\mu_1 v_3) S_\beta S_\gamma\\
&\quad - \bar\beta(\bar\gamma^2\mu_1\delta_2 + \bar\alpha^2\mu_2 v_1) S_\gamma S_{h\alpha} - \bar\gamma(\bar\beta^2\mu_3\delta_2 - \bar\alpha^2\mu_2 v_3) S_\beta S_{h\alpha}\\
\Phi_7 &= \bar\beta\mu_3\varepsilon_1 C_\beta S_\gamma S_{h\alpha} + \bar\alpha\mu_2\varepsilon_1 S_\beta S_\gamma C_{h\alpha} + \bar\gamma\mu_1\varepsilon_1 S_\beta C_\gamma S_{h\alpha}\\
\Phi_8 &= (\bar\alpha^2\mu_2\zeta_2 - \bar\beta^2\mu_3\eta_3 + \bar\gamma^2\mu_1\eta_1) S_\beta S_\gamma S_{h\alpha} + \bar\alpha\bar\gamma(\mu_2\eta_1 - \mu_1\zeta_2) S_\beta(1 - C_\gamma C_{h\alpha})\\
&\quad - \bar\alpha\bar\beta(\mu_2\eta_3 + \mu_3\zeta_2) S_\gamma(1 - C_\beta C_{h\alpha}) - \bar\beta\bar\gamma(\mu_1\eta_3 - \mu_3\eta_1) S_{h\alpha}(1 - C_\beta C_\gamma)\\
\Phi_9 &= -\varepsilon_1(\bar\beta\mu_3 S_\gamma S_{h\alpha} + \bar\alpha\mu_2 S_\beta S_\gamma + \bar\gamma\mu_1 S_\beta S_{h\alpha})\\
\Phi_{10} &= \varepsilon_1\{\bar\alpha\bar\beta S_\gamma(C_\beta - C_{h\alpha}) + \bar\beta\bar\gamma S_{h\alpha}(C_\gamma - C_\beta) + \bar\alpha\bar\gamma S_\beta(C_{h\alpha} - C_\gamma)\}\\
\Phi_{11} &= -(\bar\alpha\xi_1\mu_2 - \bar\gamma\xi_2\mu_1) C_\beta S_\gamma S_{h\alpha} + (\bar\beta\xi_1\mu_3 + \bar\gamma\xi_3\mu_1) S_\beta S_\gamma C_{h\alpha}\\
&\quad + (\bar\beta\xi_2\mu_3 - \bar\alpha\xi_3\mu_2) S_\beta C_\gamma S_{h\alpha} - (\varepsilon_3 C_\beta C_\gamma C_{h\alpha} + \bar\kappa_1 C_{h\alpha} + \bar\kappa_2 C_\beta + \bar\kappa_3 C_\gamma)\\
\Phi_{12} &= (\bar\alpha\xi_1\mu_2 - \bar\gamma\xi_2\mu_1) S_\gamma S_{h\alpha} - (\bar\beta\xi_1\mu_3 + \bar\gamma\xi_3\mu_1) S_\beta S_\gamma - (\bar\beta\xi_2\mu_3 - \bar\alpha\xi_3\mu_2) S_\beta S_{h\alpha}\\
&\quad - (\varepsilon_3 + \bar\kappa_2 C_\gamma C_{h\alpha} + \bar\kappa_3 C_\beta C_{h\alpha} + \bar\kappa_1 C_\beta C_\gamma)
\end{aligned}\right\}, \tag{52}$$

and

$$\Delta = (\bar{\alpha}^2\mu_2^2-\bar{\beta}^2\mu_3^2-\bar{\gamma}^2\mu_1^2)S_\beta S_\gamma S_{h\alpha}-2\lambda_3 S_\gamma(1-C_\beta C_{h\alpha})-2\lambda_1 S_{h\alpha}(1-C_\beta C_\gamma) -2\lambda_2 S_\beta(1-C_\gamma C_{h\alpha}), \quad (53)$$

Equations (51), together with eqns (39)–(50) and eqns (52)–(53), give all the terms of the dynamic stiffness matrix **K** of eqns (36)–(38). Note that none of these terms are zero but [see eqns (51)] that the terms $K_{1,3}$, $K_{1,6}$, $K_{2,3}$, $K_{2,6}$, $K_{3,4}$, $K_{3,5}$, $K_{4,6}$ and $K_{5,6}$ reduce to zero when $x_\alpha = 0$, i.e. when the shear and the mass centres of the beam cross-section are coincident. The dynamic stiffness matrix is then that for an axially loaded Timoshenko beam, i.e. an axially loaded Bernoulli–Euler beam with effects of shear deformation and rotatory inertia included (Howson and Williams, 1973) because $x_\alpha = 0$ can be substituted in the derived expressions without causing any overflow or underflow. In computing the dynamic stiffness matrix, the terms corresponding to the effects of axial force, shear deformation and rotatory inertia (p^2, s^2 and r^2, respectively) can optionally be made zero, with x_α non-zero, to give stiffnesses identical to, respectively, those given by the earlier coupled bending-torsional theories for a Timoshenko beam (Banerjee and Williams, 1992), an axially loaded Bernoulli–Euler beam (Banerjee and Fisher, 1992) and a Bernouli–Euler beam (Banerjee, 1989, 1991) with non-coincident mass and shear centre. Additionally, when $x_\alpha = 0$, $p^2 = 0$, $s^2 = 0$ and $r^2 = 0$, the computed dynamic stiffness matrix of eqns (51) gives the same stiffnesses as those of a Bernoulli–Euler beam (Williams and Wittrick, 1973).

3. APPLICATION OF THE DYNAMIC STIFFNESS MATRIX

The dynamic stiffness matrix terms can now be used to compute coupled bending-torsional natural frequencies and mode shapes of axially loaded Timoshenko beams, or of structures constructed from such beams. Applications include free vibration analysis of helicopter and wind turbine blades and of aircraft wings because these are all essentially an assembly of such beams. Natural frequencies are calculated by applying the well-known algorithm of Wittrick and Williams (1971), the main features of which are usually used when applying the dynamic stiffness matrix method or its buckling equivalent, e.g. see Williams and Wittrick (1983), Friberg (1983, 1985) and Oliveto (1992). Basically the algorithm needs the dynamic stiffness matrix of individual members in the structure and information about their clamped–clamped (i.e. built-in at both ends) natural frequencies. The clamped–clamped natural frequency information is needed to enable the algorithm to ensure that no natural frequencies of the structure that contains the member are missed. Thus an explicit expression from which the clamped–clamped natural frequencies can be found facilitates an easy and straight-forward application of the algorithm and hence reduces the computational efforts which would otherwise have been required (Friberg, 1985; Oliveto, 1992). Δ in eqn (53) is such an expression because the clamped–clamped natural frequencies are given by its zeros or, better still, it can be used to give the needed information about those natural frequencies without actually finding them (Banerjee and Williams, 1985). The application of the Wittrick–Williams algorithm is very simple (Banerjee, 1989) once the dynamic stiffness matrix of a structure and also an expression for clamped–clampled natural frequencies (or a method for isolating them) of its constituent members are known, but for a detailed insight see the original work of Wittrick and Williams (1971).

4. RESULTS

Numerical results were obtained using an illustrative example of a bending-torsion coupled beam of length 0.82 m and with a monosymmetric semi-circular cross-section for which comparative results are available in the literature (Friberg, 1985). Figure 4 gives the cross-sectional and material properties of this beam. The section shape factor k was calculated to be 0.5 using equation (7) of Jensen (1983), also see the classic paper by Cowper (1966).

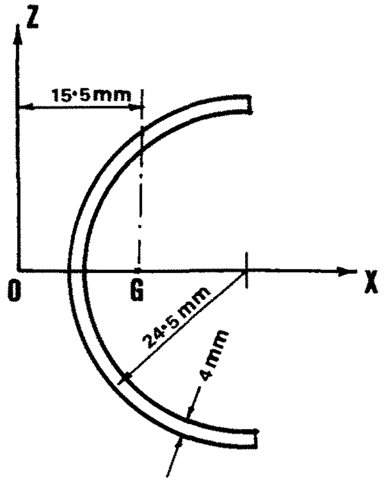

Fig. 4. Cross-sectional and material properties of a bending-torsion coupled beam (Friberg, 1985). Cross-sectional and material properties used by Friberg, which only approximately match the dimensions shown in the figure: $A = 3.08 \times 10^{-4}\ \mathrm{m}^2$; $I = I_{xx} = 9.26 \times 10^{-8}\ \mathrm{m}^4$; $m = 0.835\ \mathrm{kg\,m}^{-1}$; $J = 1.64 \times 10^{-9}\ \mathrm{m}^4$; $I_\alpha = 0.000501\ \mathrm{kg\,m}$; $E = 68.9 \times 10^9\ \mathrm{N\,m}^{-2}$; $G = 26.5 \times 10^9\ \mathrm{N\,m}^{-2}$; $\rho = 2711.04\ \mathrm{kg\,m}^{-3}$.

First, for a number of frequencies, the stiffnesses were computed numerically, i.e. the matrix inversion and matrix multiplication steps of eqn (38) were performed numerically. These stiffnesses were then compared with those given by the explicit expressions of eqns (51) and agreed with them to machine accuracy. Representative results for the 12 independent stiffness terms are given in Table 1 for two frequency values when the shear deformation, rotatory inertia and an axial force $P = 1790$ N were all present. The stiffnesses were computed using double precision arithmetic and are presented to 10 significant figures as a check for readers wishing to program the stiffness expressions themselves or to compare with other methods.

Numerical checks on calculated stiffnesses confirmed that the expressions gave correct values of stiffnesses when any combination of x_α, p^2, r^2 and s^2 were zero, i.e. when the bending-torsion coupling effect was ignored and/or axial force was ignored and/or rotatory inertia was ignored and/or shear deformation was ignored. Thus, for instance, complete

Table 1. Numerical values of dynamic stiffness matrix terms of the axially loaded bending-torsion coupled Timoshenko beam of Fig. 4 with the effects of shear deformation, rotatory inertia and axial force included, $L = 0.82$ m, $x_\alpha = 0.0155$ m, $k = 0.5$ and $P = 1790$ N

Stiffness terms	Numerical values	
	$\omega = 550$ rads^{-1}	$\omega = 1250$ rads^{-1}
$K_{1,1} = K_{4,4} =$	47489.12228	94652.95948
$K_{1,2} = -K_{4,5} =$	44906.41961	72608.07956
$K_{1,3} = K_{4,6} =$	1422.923464	−7479.574971
$K_{1,4} =$	−166077.7269	131634.3287
$K_{1,5} = -K_{2,4} =$	61830.34561	15322.40902
$K_{1,6} = K_{3,4} =$	681.5802780	−12025.36011
$K_{2,2} = K_{5,5} =$	28704.57556	34354.98588
$K_{2,3} = -K_{5,6} =$	178.8107864	−1655.396736
$K_{2,5} =$	16233.59947	8847.142820
$K_{2,6} = -K_{3,5} =$	132.0528318	−1989.792990
$K_{3,3} = K_{6,6} =$	0.909440500	309.1646998
$K_{3,6} =$	−80.84869150	346.0037922

Table 2. Coupled bending-torsional natural frequencies of the axially loaded beam of Fig. 4 (Friberg, 1985) with cantilever end conditions

Frequency number	Natural frequency (Hz) with $p^2 \neq 0$ (loaded beam) Vlasov theory (Friberg, 1985)	Present theory $r^2 \neq 0; s^2 = 0$	Present theory $r^2 \neq 0; s^2 \neq 0$
1	61.28	60.21	59.97
2	136.0	128.3	128.1
3	274.9	257.9	256.0
4	478.5	415.1	413.1

agreement was found when the stiffnesses were calculated numerically using the degenerated cases of the coupled bending-torsion Timoshenko beam (Banerjee and Williams, 1992), the coupled bending-torsion axially loaded Bernoulli–Euler beam (Banerjee and Fisher, 1992), the coupled bending-torsion Bernoulli–Euler beam (Banerjee, 1989) and the usual uncoupled Bernoulli–Euler beam (Williams and Wittrick, 1973).

Next, the first four natural frequencies with cantilever end conditions were calculated with an axial load $P = 1790$ N (which is about 40% of the lowest uncoupled Euler critical buckling load of the cantilever) acting through the centroid of the beam. The results are shown in Table 2 alongside the results of Friberg (1985). Friberg's theory neglects shear deformation, and so his results should be compared with those with $s^2 = 0$ shown when using the present theory. As expected the natural frequencies of column 3 are lower than those of Friberg, i.e. column 2. The differences between these results can probably be attributed to the fact that the present theory discounts the effect of the warping stiffness of the beam whereas Friberg's theory accounts for it. For closed sections such as solid or thin-walled aerofoils, the effect is expected to be much less pronounced. The results showing the effects of shear deformation, rotatory inertia and axial force are given in column 4 for completeness. As can be seen, the shear deformation did not make much difference for this particular problem investigated.

5. CONCLUSIONS

Explicit expressions for the exact dynamic stiffness matrix terms of an axially loaded uniform bending-torsion coupled beam have been derived taking into account the effects of shear deformation and rotatory inertia. The symbolic computing package REDUCE has been used in deriving and simplifying the stiffness terms. An expression for calculation (or identification) of the clamped–clamped natural frequencies of the beam has also been given to enable an established algorithm to be used for free vibration analysis of structures consisting of such beams. The correctness of the derived expressions has been confirmed by numerical results.

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APPENDIX

The governing partial differential equations of motion for the coupled bending-torsional free vibration of an axially loaded Timoshenko beam given by eqns (1)–(3) can be derived using Hamilton's principle as follows.

The total potential energy V of the beam of Fig. 1 is given by

$$V = \frac{1}{2}\int_0^L [EI(\theta')^2 - P\{(h')^2 - 2x_\alpha h'\psi' + (I_\alpha/m)(\psi')^2\} + kAG(h'-\theta)^2 + GJ(\psi')^2]\,\mathrm{d}y, \tag{A1}$$

where all the terms and symbols are defined in the paragraphs around eqns (1)–(3).

The total kinetic energy T is given by

$$T = \frac{1}{2}\int_0^L [m(\dot{h}^2 - 2x_\alpha \dot{h}\dot{\psi}) + I_\alpha(\dot{\psi})^2 + \rho I(\dot{\theta})^2]\,\mathrm{d}y. \tag{A2}$$

Hamilton's principle states that if $L = T - V$, where L is defined as the Lagrangian (kinetic potential), then $\int_{t_1}^{t_2} L\,\mathrm{d}t$ taken between any arbitrary intervals of time (t_1, t_2), is stationary for a dynamic trajectory. Therefore,

$$\delta \int_{t_1}^{t_2} (T - V)\,\mathrm{d}t = 0. \tag{A3}$$

Substituting eqns (A1) and (A2) into eqn (A3) gives

$$\delta \int_{t_1}^{t_2} \int_0^L \frac{1}{2} [\{m(\dot{h})^2 - 2mx_\alpha \dot{h}\dot{\psi} + I_\alpha(\dot{\psi})^2 + \rho I(\dot{\theta})^2\} - \{EI(\theta')^2 - P(h')^2 + 2Px_\alpha h'\psi' - (PI_\alpha/m)(\psi')^2 + kAG(h'-\theta)^2 + GJ(\psi')^2\}] \, dy \, dt = 0,$$

from which eqns (1)–(3) follow.